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CONTROLLABILITY TO TRAJECTORIES FOR SOME PARABOLIC SYSTEMS OF THREE AND TWO EQUATIONS BY ONE CONTROL FORCE

ASSIA BENABDALLAH *, MICHEL CRISTOFOL †, PATRICIA GAITAN ‡, AND LUZ DE TERESA §

Abstract. We present a controllability result for a class of linear parabolic systems of 3 equations. To prove the result, we establish a global Carleman estimate for the solutions of a system of 2 coupled parabolic equations with first order terms. We also obtain stability results for the identification of coefficients of the systems.

Key words. Control, observability, Carleman estimates, reaction-diffusion systems, inverse problems.

AMS subject classifications. 93B05, 93B07, 35K05, 35K55, 35R30.

1. Introduction and notations.

The controllability of linear ordinary differential systems is a well understood subject. In particular, if $n, m \in \mathbb{N}$ with $n, m \geq 1$ and $A \in \mathcal{L}(\mathbb{R}^n)$ and $C \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$. Let us define the controllability matrix

$$[A \,|\, C] = \left[A^{n-1}C \,|\, A^{n-2}C \,|\, \cdots \,|\, C\right]. \tag{1.1}$$

Then, the linear ordinary differential system $Y' = AY + Cu$ is controllable at time $T > 0$ *if and only if* the Kalman rank condition

$$\text{rank } [A \,|\, C] = n,$$

is satisfied (see for example [24, Chapter 2, p. 35]).

In the last ten years there has being an increasing interest in the study of null controllability and inverse problems for systems of parabolic equations. To the best of our knowledge most of the existing results in the litterature deal with zero order coupling systems or constant coefficients (see for instance [29], [1] [9], [2], [3], [13], [19], [21], [4], [5], [8] and [20]). In these papers, almost all the results have been established for 2×2 systems where the control is exerted on the first equation. The most general results in this context seem to be those in [20], [4] and [5]. In [20], the authors study a *cascade* parabolic system of n equations ($n \geq 2$) controlled with one single distributed control. In [4] and [5], the authors provide necessary and sufficient conditions for the controllability of $n \times n$ parabolic linear systems with constant or time-dependent coefficients. Recent results in the controllability or observability of systems show the complexity of controlling coupled parabolic equations and also the very different behavior with respect to the scalar case. For example the results in [16], [6] and [2] show that boundary controllability and distributed controllability for coupled systems are not equivalent as in the scalar case. For a complete description of the complexity of the situation and the survey of the recent results see [7].

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The main goal of our work is to propose a Carleman inequality for 3×3 parabolic linear systems with *non constant* coefficients by *one observation*. This allows us to exhibit sufficient conditions for the null controllability of the system with one control. Also this gives stability estimates for the identification of coefficients of the system. The controllability results generalize the Kalman condition obtained for parabolic systems with constant coefficients in [4]. The main ingredient of the proof is a Carleman estimate for a 2×2 reaction-convection-diffusion system.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$ be a bounded connected open set of class C^2. Let $T > 0$ and let ω be an open non empty subset of Ω. We define $\Omega_T = \Omega \times (0,T)$ and $\Sigma_T = \partial\Omega \times (0,T)$. More generally for any set $\mathcal{O} \subset \Omega$ or $\mathcal{O} \subset \partial\Omega$, $\mathcal{O}_T$ and $1_{\mathcal{O}}$ will denote resp. the set $\mathcal{O} \times (0,T)$ and the characteristic function of the set $\mathcal{O}$. Our main objective is to establish new controllability results for coupled parabolic equations with *one distributed* control force in ω_T. To be more precise, let us consider second order elliptic self adjoint operators given by

$$div(H_l\nabla) = \sum_{i,j=1}^{n} \partial_i \left(h_{ij}^l(x,t)\partial_j\right), \quad l = 1,2,3, \tag{1.2}$$

with

$$\begin{cases} h_{ij}^l \in W^{1,\infty}(\Omega_T), \\ h_{ij}^l(x,t) = h_{ji}^l(x,t) \quad \text{a.e. in } \Omega_T, \end{cases}$$

and the coefficients h_{ij}^l satisfying the uniform elliptic condition

$$\sum_{i,j=1}^{n} h_{ij}^l(x,t)\xi_i\xi_j \geq h_0|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \quad \text{a.e. in } \Omega_T, \quad l = 1,2,3, \tag{1.3}$$

for a positive constant h_0. Let $(a_{ij})_{1\leq i,j\leq 3} \in C^4(\overline{\Omega_T})$ (this assumption can be weakened see Remark 3.5). We consider the following 3×3 reaction-diffusion system

$$\begin{cases} \partial_t y = (\mathcal{L} + A)y + Cf1_\omega \text{ in } \Omega_T, \\ y = 0 \text{ on } \Sigma_T, \\ y(\cdot,0) = y_0(\cdot) \text{ in } \Omega, \end{cases} \tag{1.4}$$

where

$$\mathcal{L} = \begin{pmatrix} div(H_1\nabla) & 0 & 0 \\ 0 & div(H_2\nabla) & 0 \\ 0 & 0 & div(H_3\nabla) \end{pmatrix},$$

$A = (a_{ij})_{1\leq i,j\leq 3}$, $C = (1,0,0)^t \in \mathcal{L}\left(\mathbb{R},\mathbb{R}^3\right)$, $f \in L^2(\Omega_T)$ and $y_0 = (y_{0,i})_{1\leq i\leq 3} \in L^2(\Omega)^3$. In (1.4), $y = (y_i)_{1\leq i\leq 3}$ is the state variable.

We will say that (1.4) is null controllable if for all initial data in $L^2(\Omega)^3$ there exists $f \in L^2(\Omega_T)$ such that the solution of (1.4) satisfy $y(\cdot,T) = 0$. In [4] it has been proved, among other results, that if all the coefficients (a_{ij}) are *constant* and if there exists $d_1 \in \mathbb{R}$ such that $d_1H_1 = H_2 = H_3$, then system (1.4) is null controllable if and only if the algebraic Kalman condition, $\det[A\,|\,C] \neq 0$, is satisfied (here $[A\,|\,C]$ is given by (1.1)). On the other hand, the results in [20] are valid for non constant coefficients in A but the matrix A is *in cascade*. That means that a prefixed structure of A is required. Our main interest is to remove the assumption that the coefficients have to be constant or A to be a cascade matrix.

The main result of this paper is

THEOREM 1.1. *Suppose that there exists $j \in \{2,3\}$ such that $|a_{j1}(x,t)| \geq C > 0$ for all $(x,t) \in \omega_T$ and that $H_2 = H_3$. Let j be as before. We define $k_j = \frac{6}{j}$,*

$$B_{k_j} := -2H_2 \left(\nabla a_{k_j 1} - \frac{a_{k_j 1}}{a_{j1}} \nabla a_{j1} \right),$$

and

$$b_j = \frac{2H_2 \nabla a_{j1}(\nabla a_{k_j 1} a_{j1} - \nabla a_{j1} a_{k_j 1})}{a_{j1}^2} + \frac{a_{k_j 1} div(H_2 \nabla a_{j1}) - a_{j1} div(H_2 \nabla a_{k_j 1})}{a_{j1}}$$

$$+ \frac{a_{j1}\partial_t a_{k1} - a_{k1}\partial_t a_{j1}}{a_{j1}} - (-1)^j \frac{\det [A \,|\, C]}{a_{j1}}.$$

Assume that $\partial\omega \cap \partial\Omega = \gamma$, $|\gamma| \neq 0$, and $B_{k_j} \cdot \nu \neq 0$, on γ_T .

Then, for all $y_0 \in (L^2(\Omega))^3$, there exists $f \in L^2(\Omega_T)$ such that the corresponding solution of (1.4) satisfies $y_1(\cdot,T) = y_2(\cdot,T) = y_3(\cdot,T) = 0$.

REMARK 1.2.

1. *If all the coefficients are constant, $B_{k_j} = 0$ for all $j \in \{2,3\}$, then assumptions of Theorem 1.1 are reduced to $b_j \neq 0$ and more precisely to $\det[A|C] \neq 0$. So we recover the condition obtained in [5].*
2. *The result is valid if $B_{k_j}(x,t) = 0$ in ω'_T for some open set $\omega' \subset \omega$, and if we assume that $|b_j(x,t)| \geq \alpha > 0$ in ω'_T for some $\alpha > 0$.*
3. *It is not difficult to construct an example where the algebraic Kalman rank condition is not satisfied but assumptions of Theorem 1.1 is verified. For example, take Ω any smooth domain in $\mathbb{R}^2$ containing $\omega = \{y < -1, (x-2)^2 + (y+1)^2 < 1\}$ and $\gamma = [1,3] \times \{-1\}$. Take now $a_{32} = a_{23} = a_{22} = a_{33} = 0$, $a_{31}(x,y) = -y^2$, $a_{21}(x,y) = x + y$ and $H_1 = H_2 = H_3 = I_d$. Then $a_{31} \neq 0$ in ω_T, but $\det[A,C] = 0$ in Ω.*
   Moreover, we have
   $$\left(\nabla a_{21} - \frac{a_{21}}{a_{31}} \nabla a_{31} \right) \cdot \nu(x) = 2x - 1 > 0, \quad \text{on } \gamma.$$
   So the assumptions of Theorem 1.1 are satisfied.
4. *In our proof the fact that $\partial\omega \cap \partial\Omega = \gamma$ is used to invert a suitable operator and it is an open question if this is only a technical requirement or not.*

The proof of Theorem 1.1 relies in an essential way in the obtention of a new Carleman inequality for two coupled reaction-diffusion-convection equations. This inequality uses Carleman estimates for *scalar* parabolic equations introduced by Fursikov and Imanuvilov in [18] (see also Theorem 2.4) but is new in the sense that we are working with a system of equations so in particular new assumptions are required in the control region and the kind of coupling.

Also it is important to remark that even if the technique used to obtain this inequality is based in the technique used in [8], the inequality given in [8] is improved in the present paper with the consequence that results in [8] imply *approximate controllability* of the 2x2 reaction-diffusion system (2.1) and our results imply *null controllability* of the same system.

Our controllability result does not need restrictive assumptions such as for *cascade* systems or constant coefficients. In the case of constant coefficients, we recover the Kalman's criterion proved in [4]. In Section 2, we derive a controllability result for a class of 2×2 reaction-diffusion-convection systems. In [21], the author studies the null controllability of systems of two parabolic equations, where the coupling terms are first order space derivatives in one equation and second order space derivatives in the other. The results of [21] cannot be derived as a particular case of ours. The reverse holds true as well. They are independent results, proved with different techniques, both aiming for a better understanding of the complexity of Carleman estimates for coupled equations and their controllability and identification consequences.

A large area of applications in ecology and biology requires the identification of coefficients for reaction-diffusion systems. Starting with the pioneer work of Bukhgeim-Klibanov [10], Carleman estimates have been successfully used for the uniqueness and stability for determining coefficients. Often, it is difficult to observe all the components for reaction-diffusion systems, thus the increasing interest in reducing the number of observed components. There are very few research papers devoted to this problem. We can refer to [13], [8], for 2×2 parabolic systems and to [23] for Lamé sytems. We use our new Carleman estimate for 2×2 parabolic systems with advection terms to prove, for a 3×3 reaction-diffusion system, a stability result for three coefficients (one in each equation). We only observe one component and we need the knowledge of the solution at a fixed time $T'\in(0,T)$ and all the domain Ω and these three coefficients in ω. We generalize for a $n\times n$ parabolic system: a stability result for n coefficients observing only $n-2$ components with the partial knowledge of solely three coefficients.

The paper is organized as follows. In Section 2 we prove a new controllability result for a 2×2 reaction-diffusion-convection system. This section contains the main ingredient of this paper: a new Carleman estimate for 2×2 reaction-diffusion-convection systems with only one observation. In Section 3 we prove our main result: Theorem 1.1. To prove this, we establish an observability estimate for the corresponding adjoint system. This Carleman estimate for 3×3 reaction-diffusion systems follows from the previous one for 2×2 reaction-diffusion-convection systems. We conclude in Section 4 with some applications to inverse problems and generalizations for more than 3 equations. In Section 5 we include some comments related with other kind of parabolic systems and propose some open problems. We conclude with an Appendix where we explain the change of coordinates that allows to simplify the proof of the Carleman estimate for the 2×2 reaction-diffusion-convection systems.

2. Controllability for a 2×2 reaction-diffusion-convection system. In this section we present a null controllability result for coupled parabolic systems. We consider the case of two coupled reaction-diffusion-convection equations and we control one of them. Let us consider

$$\left\{\begin{array}{lll}\partial_t y_1 = div(H_1\nabla y_1) + a_{11}\ y_1 + a_{12}\ y_2 + A_{11}\cdot\nabla y_1 + A_{12}\cdot\nabla y_2 + f\chi_\omega & \text{in} & \Omega_T,\\ \partial_t y_2 = div(H_2\nabla y_2) + a_{21}\ y_1 + a_{22}\ y_2 + A_{21}\cdot\nabla y_1 + A_{22}\cdot\nabla y_2 & \text{in} & \Omega_T,\\ y_1(\cdot,t) = y_2(\cdot,t) = 0 & \text{on} & \Sigma_T,\\ y_1(\cdot,0) = y_1^0(\cdot),\ y_2(\cdot,0) = y_2^0(\cdot) & \text{in} & \Omega.\end{array}\right. \tag{2.1}$$

with $a_{ij} \in C^4(\overline{\Omega_T})$ and $A_{ij} \in C^1(\overline{\Omega_T})^n$ for $i, j = 1, 2$ (here again this regularity can be weakened see Remark 3.5). We say that system (2.1) is null controllable if for every initial data $(y_1^0, y_2^0) \in (L^2(\Omega))^2$ there exists $f \in L^2(\omega_T)$ such that the corresponding solution of (2.1) satisfies

$$y_1(T) = y_2(T) = 0.$$

The first controllability result we obtain is the following one :

THEOREM 2.1. *Let us assume that ω of class C^2, $\omega \subset \Omega$ is such that for some $\gamma \subset \partial\Omega$, $|\gamma| \neq 0$ with $\gamma \subset \partial\omega \cap \partial\Omega$ we have that $|A_{21}(x,t) \cdot \nu(x)| \neq 0$ for every $(x,t) \in \gamma_T$. Furthermore, assume that $H_1|_{\omega_T} \in W^{2,\infty}(\omega_T)^{n^2}$ and that $A_{21}|_{\omega_T} \in W^{3,\infty}(\omega_T)^n$. Then, system (2.1) is null controllable at time $T > 0$.*

REMARK 2.2. *As in Theorem 1.1 the result is valid if $A_{21}(x,t) = 0$ in ω'_T for some open set $\omega' \subset \omega$, and if we assume that $|a_{21}(x,t)| \geq \alpha > 0$ in ω'_T for some $\alpha > 0$.*

As in the scalar case the null controllability of (2.1) is equivalent to the obtention of an observability inequality for the adjoint system (2.3) (see e.g [18], [15], [14]). That is, Theorem 2.1 is equivalent to prove the following result:

THEOREM 2.3. *Under the assumptions of Theorem 2.1 there exists $C > 0$ such that*

$$\int_\Omega \left(|\varphi_1(0)|^2 + |\varphi_2(0)|^2\right) dx \leq C \int_0^T \int_\omega |\varphi_1|^2 dxdt \tag{2.2}$$

holds true for any solution of

$$\begin{cases} -\partial_t\varphi_1 = div(H_1\nabla\varphi_1) + (a_{11} - \nabla \cdot A_{11})\varphi_1 & \\ \qquad +(a_{21} - \nabla \cdot A_{21})\varphi_2 - A_{11} \cdot \nabla\varphi_1 - A_{21} \cdot \nabla\varphi_2 & in \quad \Omega_T, \\ -\partial_t\varphi_2 = div(H_2\nabla\varphi_2) + (a_{12} - \nabla \cdot A_{12})\varphi_1 & \\ \qquad +(a_{22} - \nabla \cdot A_{22})\varphi_2 - A_{12} \cdot \nabla\varphi_1 - A_{22} \cdot \nabla\varphi_2 & in \quad \Omega_T, \\ \varphi_1(\cdot,t) = \varphi_2(\cdot,t) = 0 & on \quad \Sigma_T, \\ \varphi_1(\cdot,T) = \varphi_1^T(\cdot), \; \varphi_2(\cdot,T) = \varphi_2^T(\cdot) & in \quad \Omega. \end{cases} \tag{2.3}$$

2.1. Proof of the controllability result. We prove Theorem 2.1 assuming that Theorem 2.3 holds true. The proof of Theorem 2.3 uses the Carleman inequality proved in the next subsection. So we prove Theorem 2.3, and conclude the section with the proof of the Carleman inequality, i.e Theorem 2.6.

Proof. [Proof of Theorem 2.1]

We now prove Theorem 2.1 using (2.2). They are several ways to prove it. We use the most direct technique. Let $V = L^2(\Omega) \times L^2(\Omega)$, and let G and L be the following linear mappings:

$$L : L^2(\omega_T) \to V \qquad\qquad f \mapsto (y_1(T), y_2(T))$$

where $(y_1(\cdot), y_2(\cdot))$ is the corresponding solution of (2.1) with $(y_1^0, y_2^0) = (0, 0)$, and

$$G : V \to V \qquad\qquad (y_1^0, y_2^0) \mapsto (y_1(T), y_2(T))$$

where $(y_1(\cdot), y_2(\cdot))$ solves (2.1) with $f = 0$. Then Theorem 2.1 is equivalent to the inclusion

$$R(G) \subset R(L). \tag{2.4}$$

Both G and L are V-valued, bounded linear operators. So (2.4) holds if and only if, for every $(\varphi_1^T, \varphi_2^T) \in V$,

$$\|G^*(\varphi_1^T, \varphi_2^T)\|_V \le C \|L^*(\varphi_1^T, \varphi_2^T)\|_{L^2(\omega_T)} \tag{2.5}$$

for some constant $C > 0$. A simple computation shows that

$$G^*(\varphi_1^T, \varphi_2^T) = (\varphi_1(x,0), \varphi_2(x,0)), \quad L^*(\varphi_1^T, \varphi_2^T) = \varphi_1 1_{\omega_T}$$

where φ_1 and φ_2 solve the adjoint system (2.3). Hence (2.5) is just (2.2) and Theorem 2.1 is proved. □

2.2. A new Carleman estimate for a 2×2 reaction-diffusion-convection system with one observation. In this section we will prove a Carleman estimate for a 2×2 reaction-diffusion-convection system. To this end we need to recall Carleman's estimates for a scalar equation. We will use the classical notations (see [18] and [22]).

Let $\beta \in C^2(\overline{\Omega})$ be the function constructed by Fursikov and Imanuvilov in [18] such that $\beta \ge 0$ in $\bar{\Omega}$, $|\nabla\beta| > 0$ in $\Omega \setminus \overline{\omega}$ and define

$$\begin{cases} \eta(x,t) & := \dfrac{e^{2\lambda K} - e^{\lambda\beta(x)}}{t(T-t)}, & \forall (x,t) \in \Omega_T, \\ \rho(x,t) & := \dfrac{e^{\lambda\beta(x)}}{t(T-t)}, & \forall (x,t) \in \Omega_T, \end{cases} \tag{2.6}$$

where

$$K \ge \|\beta\|_{L^\infty(\Omega)} \tag{2.7}$$

is a fixed constant whose choice will be specified later. We introduce the functional

$$I(\tau,\varphi) = \iint_{\Omega_T} (s\rho)^{\tau-1} e^{-2s\eta} \left(|\varphi_t|^2 + \sum_{1\le i\le j\le n} \left|\partial^2_{x_i x_j}\varphi\right|^2 + (s\lambda\rho)^2 |\nabla\varphi|^2 + (s\lambda\rho)^4 |\varphi|^2 \right) dxdt \tag{2.8}$$

where $s, \lambda > 0$ and $\tau \in \mathrm{I\!R}$.

For $a \in L^\infty(\Omega_T)$, $A \in L^\infty(\Omega_T)^n$, $H \in W^{1,\infty}(\Omega_T)^{n^2}$, let $R = div(H\nabla) + A\cdot\nabla + a$ and $I(\tau,\varphi)$ defined by (2.8) we have the following result

THEOREM 2.4. *Let $\omega \subset \Omega$ open and non empty, $\tau \in \mathrm{I\!R}$. Then, there exist three positive constants s_0, λ_0, C_0 (which only depend on Ω, ω, T, $\|H\|_{W^{1,\infty}}$, $\|A\|_{L^\infty}$, $\|a\|_{L^\infty}$ and τ) such that for every $\varphi \in L^2(0,T;H_0^1(\Omega))$ with $\partial_t\varphi \pm R\varphi \in L^2(\Omega_T)$, the following Carleman estimate holds*

$$I(\tau,\varphi) \le \widetilde{C}_0 \left(\iint_{\Omega_T} (s\rho)^\tau e^{-2s\eta} |\partial_t\varphi \pm R\varphi|^2 \, dxdt + \lambda^4 \iint_{\omega_T} (s\rho)^{\tau+3} e^{-2s\eta} |\varphi|^2 \, dxdt \right),$$

for all $s \ge s_0$, $\lambda \ge \lambda_0$ and η, ρ defined in (2.6) with $K > 0$ satisfying (2.7). The proof of this result can be found in [22].

We consider the following reaction-diffusion-convection system:

$$\begin{cases} \partial_t u = div(H_1\nabla u) + au + bv + A\cdot\nabla u + B\cdot\nabla v + f & \text{in} \quad \Omega_T, \\ \partial_t v = div(H_2\nabla v) + cu + dv + C\cdot\nabla u + D\cdot\nabla v + g & \text{in} \quad \Omega_T, \\ u(\cdot,t) = v(\cdot,t) = 0 & \text{on} \quad \Sigma_T, \\ u(\cdot,0) = u_0(\cdot), v(\cdot,0) = v_0(\cdot) & \text{in} \quad \Omega. \end{cases} \tag{2.9}$$

Recall that for $a, b, c, d \in L^\infty(\Omega_T)$ and $A, B, C, D \in L^\infty(\Omega_T)^n$, $H_1, H_2 \in W^{1,\infty}(\Omega_T)^{n^2}$ and for $u_0, v_0 \in L^2(\Omega)$, $f, g \in L^2(\Omega_T)$ the reaction-diffusion-convection system (2.9) admits an unique solution $(u, v) \in C([0,T]; L^2(\Omega))^2 \cap L^2(0,T; H_0^1(\Omega))^2$. Moreover, if $u_0, v_0 \in H^2(\Omega)\cap H_0^1(\Omega)$, then $(u, v) \in C([0,T]; H^2(\Omega) \cap H_0^1(\Omega))^2 \cap C^1([0,T]; L^2(\Omega))^2$. This result can be obtained in a classical way, see for example [26].

In order to prove (2.2), our main interest is to derive an observability estimate for (u, v) solutions of (2.9) by *solely* the observation of u in ω_T.

ASSUMPTION 2.5.

1. *Let $\omega \subset \Omega$ be a non-empty subdomain of class C^2 with $\partial\omega \cap \partial\Omega = \gamma$ and $|\gamma| \neq 0$,*
2. *$B \cdot \nu(x) \neq 0$, on γ_T,*
3. *$H_1|_{\omega_T} \in W^{2,\infty}(\omega_T)^{n^2}$, $B|_{\omega_T} \in W^{2,\infty}(\omega_T)^n$, $A|_{\omega_T} \in W^{1,\infty}(\omega_T)^n$ and $b|_{\omega_T} \in W^{2,\infty}(\omega_T)$.*

The new Carleman estimate for a 2×2 reaction-diffusion-convection system is:

THEOREM 2.6. *Under Assumption 2.5 there exist three positive constants s_0, λ_0, C (which only depend on Ω, ω, a, b, c, d, A, B, C, D, H_1, H_2, τ_1 and τ_2) and a constant K, satisfying (2.23), such that for every $(u_0, v_0) \in L^2(\Omega)^2$ and $|\tau_1 - \tau_2| < 1$, the following Carleman estimate holds*

$$\begin{aligned} I(\tau_1, u) \quad &+ I(\tau_2, v) \leq C \Bigg(\lambda^8 \iint_{\omega_T} e^{-2s\alpha} \ (s\rho^*)^{\tau^*} \ |u|^2 dxdt \\ &+ \lambda^4 \iint_{\omega_T} e^{-2s\eta} (s\rho)^{3+\tau_2} |Qf|^2 dxdt + \iint_{\Omega_T} e^{-2s\eta}((s\rho)^{\tau_1}|f|^2 + (s\rho)^{\tau_2}|g|^2) dxdt \Bigg) \end{aligned} \tag{2.10}$$

where Q is an appropriate operator defined in the Appendix (see (6.5)), $\eta^* = \max_{\overline{\Omega}} \eta$, $\eta_- = \min_{\overline{\Omega}} \eta$, $\alpha = 4\eta_- - 3\eta^*$, $\rho^* = \max_{\overline{\Omega}} \rho$ and $\tau^* = 4\tau_2 - 3\tau_1 + 15$, for all $s \geq s_0$, $\lambda \geq \lambda_0$ for all (u, v) solution of (2.9) and η defined by (2.6) .

REMARK 2.7. *Here it is important to recall the results in [8] where the following inequality for system (2.9) was proved.*

$$\begin{aligned} I(\tau, u) + I(\tau, v) \quad &\leq \\ &C \left(\|u\|^2_{W_2^{2,1}(\omega_T)} + \|f\|^2_{L^2(\omega_T)} + \int_{\Omega_T} (s\rho)^\tau e^{-2s\eta}(|f|^2 + |g|^2) \right) \end{aligned} \tag{2.11}$$

for all $s \geq s_0$ and where

$$W_2^{m,m/2}(\Omega_T) = \left\{ u : \Omega_T \to \mathbb{R}; \partial_x^\alpha \partial_t^{\alpha_{n+1}} u \in L^2(\Omega_T), \ for \ |\alpha| + 2\alpha_{n+1} \leq m \right\}.$$

Note that (2.11) is a weaker Carleman estimate with respect to (2.10) since the norms in the local terms of u are different. In particular the estimate (2.11) cannot imply null controllability with controls in $L^2(\omega_T)$ but just approximate controllability. Moreover the stability results in [8] obtained for the inverse problem require stronger norms for the observation. Our main contribution is then the obtention of a local L^2 norm of u in the right hand side of (2.10).

We first prove Theorem 2.3 assuming that Theorem 2.6 holds true.

Proof. [Proof of Theorem 2.3]

We want to prove (2.2). We define $u(t) = \varphi_1(T - t), v(t) = \varphi_2(T - t)$. Then, u and v are solutions of (2.9) with $a(x, t) = (a_{11}(x, T - t) - \nabla \cdot A_{11}(x, T - t)$, $b(x, t) = (a_{21}(x, T - t) - \nabla \cdot A_{21}(x, T - t))$, $A(x, t) = -A_{11}(x, t - T)$, $B(x, t) = -A_{21}(x, t - T)$, $c(x, t) = (a_{12}(x, T - t) - \nabla \cdot A_{12}(x, T - t)$, $d(x, t) = (a_{22}(x, T - t) - \nabla \cdot A_{22}(x, T - t))$,

$C(x,t) = -A_{12}(x, t-T)$, $D(x,t) = A_{22}(x, t-T)$, $f = g = 0$ and initial conditions $u(0) = \varphi_1^T$, $v(0) = \varphi_2^T$. We can then apply the results of Theorem 2.6 to φ_1 and φ_2 and get

$$\iint_{\Omega_T} e^{-2s\eta}\rho^3(|\varphi_1|^2 + |\varphi_2|^2)dxdt \le C \iint_{\omega_T} |\varphi_1|^2 dxdt.$$

Note that $C > 0$ is a generic constant that may change from line to line. We multiply the first equation in (2.3) by φ_1 and the second equation by φ_2 , we add the two equations, we integrate by parts and apply Gronwall inequality, we then get that for any $0 \le t < \tau \le T$

$$\int_\Omega (|\varphi_1(t)|^2 + |\varphi_2(t)|^2)dx \le C \int_\Omega (|\varphi_1(\tau)|^2 + |\varphi_2(\tau)|^2)dx.$$

This inequality implies on one hand that for $\tau > 0$

$$\int_\Omega (|\varphi_1(0)|^2 + |\varphi_2(0)|^2)dx \le C \int_\Omega (|\varphi_1(\tau)|^2 + |\varphi_2(\tau)|^2)dx,$$

and on the other hand that

$$\int_\Omega (|\varphi_1(0)|^2 + |\varphi_2(0)|^2)dx \le C(T) \int_{T/4}^{3T/4}\int_\Omega (|\varphi_1(\tau)|^2 + |\varphi_2(\tau)|^2)dxd\tau. \tag{2.12}$$

Observe that, by construction, $\rho^3(t)e^{-2s\eta(t)} \ge C(T)$ for $t \in [T/4, 3T/4]$. This fact combined with (2.12) imply that

$$\int_\Omega (|\varphi_1(0)|^2 + |\varphi_2(0)|^2)dx \le C \iint_{\omega_T} |\varphi_1|^2 dxdt.$$

□

Proof. [Proof of Theorem 2.6]

Consider $\widetilde{\omega} \subset\subset \omega$ open and non empty. If $|\tau_1 - \tau_2| < 1$, a direct application of Theorem 2.4 leads to

$$\begin{aligned} I(\tau_1, u) + I(\tau_2, v) \le C \Bigg(\lambda^4 \iint_{\widetilde{\omega}_T} (s\rho)^{\tau_1+3} e^{-2s\eta}|u|^2 dxdt \;+ \lambda^4 \iint_{\widetilde{\omega}_T} (s\rho)^{\tau_2+3} e^{-2s\eta}|v|^2 dxdt \\ + \iint_{\Omega_T} (s\rho)^{\tau_1} e^{-2s\eta} |f|^2 dxdt \;+ \iint_{\Omega_T} (s\rho)^{\tau_2} e^{-2s\eta} |g|^2 dxdt \Bigg). \end{aligned} \tag{2.13}$$

The main question is to get rid of the term

$$\iint_{\widetilde{\omega}_T} (s\rho)^{\tau_2+3} e^{-2s\eta}|v|^2 dxdt \quad .$$

As in ([2], [13]), one will use the first equation in order to derive a *local observability* estimate for v with respect to u. In comparison with the previous results, the main difficulty here is the presence of *first order terms* on v. Roughly speaking, the idea used in ([2], [13]) is to *transform locally (in $\omega \times (0,T)$)* the first equation of (2.9) as $v = Pu$ where P is a *partial differential operator (first order in time and second order in space)*. In these cases, $A = B = C = D = 0$ and therefore the main assumption for this solvability is that $b \neq 0$ in an open set $\omega' \times (0,T) \subset \omega_T$. In our case, this condition is replaced by $B \cdot \nu \neq 0$ on $\gamma \times (0,T)$ where γ is a part of the boundary of $\Omega \cap \omega$ and requires ω to be a neighborhood of γ. With these assumptions, one can still transform the first equation of (2.9) *locally* in space as $v = Pu$, but the operator P is not a partial differential operator. With some technical computations, we are able to still deduce a local observability estimate for v with respect to u:

THEOREM 2.8. *Under the assumptions of Theorem 2.6, suppose that* $u \in C([0,T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ *and* v *satisfies*

$$\begin{cases} bv + B \cdot \nabla v = \partial_t u - div(H_1 \nabla u) - au - A \cdot \nabla u - f & in \quad \omega_T, \\ v = 0 & on \quad \gamma_T. \end{cases}$$

Then, for any $\widetilde{\widetilde{\omega}} \subset\subset \omega$ *and for all* $\varepsilon > 0$, *there exists* $C_\varepsilon > 0$ *such that*

$$\begin{aligned} \lambda^4 \iint_{\widetilde{\widetilde{\omega}}_T} (s\rho)^{\tau_2+3} e^{-2s\eta} |v|^2 dxdt \leq C_\varepsilon \lambda^{16} \iint_{\omega_T} e^{-2s\alpha} \ (s\rho^*)^{\tau^*} |u|^2 dxdt \\ + \lambda^4 \iint_{\omega_T} e^{-2s\eta} (s\rho)^{3+\tau_2} |Qf|^2 dxdt + \varepsilon I(\tau_2, v), \end{aligned} \tag{2.14}$$

where Q is an appropriate operator defined in the Appendix (see (6.5)), $\eta^* = \max_{\overline{\Omega}} \eta$, $\eta_- = \min_{\overline{\Omega}} \eta$, $\alpha = 4\eta_- - 3\eta^*$, $\rho^* = \max_{\overline{\Omega}} \rho$ *and* $\tau^* := \max\{2\tau_2 - \tau_1 + 7, 1 - \tau_1, 4\tau_2 - 3\tau_1 + 15, \tau_2 + 3\}$, *for all* $s \geq s_0$, $\lambda \geq \lambda_0$ *and* η *defined by (2.6).*

Proof. [Proof of Theorem 2.8] In order to make the proof clearer to the reader, we are going to prove Theorem 2.8 in the simplest case where

$$\Omega_T = (0,1) \times \Omega'_T, \quad \omega_T = (0,\epsilon) \times \omega'_T, \quad \gamma = \{0\} \times \omega',$$

with $\omega' \subset \Omega' \subset \mathbb{R}^{n-1}$, $\epsilon > 0$ and $B(x,t) = (1,0,...,0,0)$.

Note that for more general vector field B we obtain, by a change of variables, a similar equation:

$$\partial_1 \widetilde{v} + b\widetilde{v} = \partial_t \widetilde{u} - div(H \nabla \widetilde{u}) - E \cdot \nabla \widetilde{u} - e\widetilde{u} - \widetilde{f}.$$

The proof of this assertion is given in Appendix 6.1 so the general case comes down to this simplest one.

With these assumptions, the first equation of (2.9) has the particular form

$$\partial_1 v + bv = \partial_t u - div(H_1 \nabla u) - au - A \cdot \nabla u - f, \tag{2.15}$$

with $x = (x_1, x')$.

The proof of the Theorem will be done in 3 steps

- Step 1 : **An equation for v**

  One can define the following operator

  $$L := \partial_1 + b,$$

  with $D(L) = \{v \in L^2(0,T); H^1(\omega)); v(0,x',t) = 0$ on $\gamma_T)$. Note that $(L, D(L))$ is an unbounded invertible operator from $L^2(\omega_T)$ to $L^2(\omega_T)$. For $w \in L^2(\omega_T)$, direct computations give that

  $$L^{-1}(w)(x,t) = e^{\int_0^{x_1} b(y_1,x',t)dy_1} \int_0^{x_1} e^{-\int_0^{y_1} b(x_1,x',t)dx_1} w(y_1,x',t)dy_1.$$

For, $p, q \in L^\infty(\omega_T)$, let us define

$$K(p,q)w(x,t) = p(x,t)\int_0^{x_1} q(y_1,x',t)w(y_1,x',t)dy_1.$$

Note that $K(p,q) \in \mathcal{L}(L^2(\omega_T))$ and $L^{-1} = K(p,q)$ with

$$p(x,t) = e^{-\int_0^{x_1} b(y_1,x',t)dy_1}, \ \ q(x,t) = e^{\int_0^{x_1} b(y_1,x',t)dy_1}. \tag{2.16}$$

Moreover under Assumption 2.5, we have $p, q \in W^{2,\infty}(\omega_T)$. As it will be clear in the sequel, we need to compute the effect of the composition of operators as $K(p,q)$ on partial differential operators.
We summarize here these computations:

1. For $p, q, e \in L^\infty(\omega_T)$, $w \in L^2(\omega_T)$

$$K(p,q)(ew) = K(p,qe)w.$$

2. For $p, q \in W^{1,\infty}(\omega_T)$ and $E \in L^\infty(0,T;W^{1,\infty}(\omega))$, $2 \le i \le n$ and $w \in H^1(\omega_T)$, we have

$$K(p,q)(E\partial_1 w)(x,t) = -K(p,\partial_1(Eq)w)(x,t) + pqEw(x,t) - p(x,t)(qEw)(0,x',t),$$

$$K(p,q)(E\partial_i w)(x,t) = \partial_i K(p,Eq)w(x,t) - K(\partial_i p, Eq)w(x,t) - K(p,\partial_i(Eq))w(x,t),$$

$$K(p,q)(\partial_t w)(x,t) = \partial_t K(p,q)w(x,t) - K(\partial_t p, q)w(x,t) - K(p,\partial_t q)w(x,t).$$

3. For $p, q \in W^{2,\infty}(\omega_T)$, $H \in L^\infty(0,T;W^{1,\infty}(\omega))$, $1 \le i,j \le n$ and $w \in L^2(0,T;H^2(\omega))$, we have

$$K(p,q)(\partial_i(h_{ij}\partial_j w))(x,t) = K(p, q\partial_i h_{ij})\partial_j w(x,t) + K(p, h_{ij}q)\partial^2_{i,j} w(x,t),$$

In order to calculate the first term in the previous equation we use the computations done in item 2. We have to develop the second term. To do this we observe that

$$K(p,q)\partial_1^2 w(x,t) = K(p,\partial_1^2 q)w(x,t) - (p\partial_1 q w)(x,t) + (pq\partial_1 w)(x,t) + p(x,t)(w\partial_1 q - q\partial_1 w)(0,x',t),$$

and for $2 \le i,j \le n$ we have that

$$\begin{aligned} K(p,q)(\partial^2_{i,j} w)(x,t) &= \partial^2_{i,j} K(p,q)w(x,t) - K(\partial_{ij} p, q)w - K(p,\partial_{ij} q)w \\ &\quad - K(\partial_i p, \partial_j q)w(x,t) - K(\partial_j p, \partial_i q)w(x,t) \\ &\quad - K(\partial_i p, q)\partial_j w(x,t) - K(\partial_j p, q)\partial_i w(x,t) - K(p,\partial_i q)\partial_j w(x,t) - K(p,\partial_j q)\partial_i w(x,t). \end{aligned}$$

Furthermore for $1 \le j \le n$

$$K(p,q)(\partial^2_{1,j} w)(x,t) = \partial_j[K(p,q)\partial_1 w(x,t)] - K(\partial_j p, q)\partial_1 w(x,t) - K(p,\partial_j q)\partial_1 w(x,t).$$

Lemma 2.9. *Let H_1, A, b, B satisfy Assumption 2.5, p, q defined in (2.16). Then there exist $(p_{ij}, q_{ij})_{2\le i,j\le n} \in W^{2,\infty}(\omega_T)^{2(n-1)^2}$, $(\tilde{p}_i, \tilde{q}_i)_{1\le i\le n} \in W^{1,\infty}(\omega_T)^{2n}$, $k \in L^\infty(\omega_T)$, $E_i \in W^{1,\infty}(\Omega_T)$, $i = 1, \cdots, n$ such that, for any $u \in C([0,T];H^2(\Omega)\cap$*

$H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$, *the solution* v *of (2.8) satisfies for every* $(x,t) \in \omega_T$ *:*

$$\begin{aligned} v(x,t) = \quad & \partial_t K(\tilde{p_1},\tilde{q_1})u(x,t) + \sum_{i,j=2}^{n} \partial_{i,j}^2 K(p_{ij},q_{ij})u(x,t) + \sum_{i=2}^{n} \partial_i K(\tilde{p_i},\tilde{q_i})u(x,t) \\ & + \sum_{i=1}^{n} \partial_i (E_i u)(x,t) + K(p,aq)u(x,t) + k(x,t)u(x,t) \\ & + K(p,q)f(x,t) - p(x,t)q(0,x',t)h_{11}^1(0,x',t)\partial_1 u(0,x',t). \end{aligned} \tag{2.17}$$

Proof. [Proof of the Lemma 2.9] It is a direct consequence of the previous items taking into account that u vanishes on γ.□

- Step 2: **An observability inequality for v with two observations: u on ω_T and $\partial_\nu u$ on $\gamma \times (0,T)$.**
Let $\widetilde{\widetilde{\omega}} \subset\subset \widetilde{\omega} \subset\subset \omega$, and $\xi \in C^\infty(\overline{\Omega})$ a cut-off function, such that

$$\xi(x) = \begin{cases} 1, \forall x \in \overline{\widetilde{\widetilde{\omega}}} \\ 0, \forall x \notin \omega. \end{cases} \tag{2.18}$$

We multiply (2.17) by $(s\rho)^{\tau_2+3}\xi e^{-2s\eta}v$ and we integrate on Ω_T.

$$\begin{aligned} & \iint_{\Omega_T} (s\rho)^{\tau_2+3}\xi e^{-2s\eta}v^2 dxdt = \iint_{\Omega_T} (s\rho)^{\tau_2+3}\xi e^{-2s\eta} v\partial_t K(\tilde{p_1},\tilde{q_1})udxdt \\ & + \iint_{\Omega_T} (s\rho)^{\tau_2+3}\xi e^{-2s\eta} v \sum_{i,j=2}^{n} \partial_{i,j}^2 K(p_{ij},q_{ij})udxdt \\ & + \iint_{\Omega_T} (s\rho)^{\tau_2+3}\xi e^{-2s\eta} v \sum_{i=2}^{n} \partial_i K(\tilde{p_i},\tilde{q_i})udxdt \\ & + \iint_{\Omega_T} (s\rho)^{\tau_2+3}\xi e^{-2s\eta} v \sum_{i=1}^{n} \partial_i (E_i u)dxdt \\ & + \iint_{\Omega_T} (s\rho)^{\tau_2+3}\xi e^{-2s\eta} vK(p,aq)udxdt + \iint_{\Omega_T} (s\rho)^{\tau_2+3}\xi e^{-2s\eta} vkudxdt \\ & + \iint_{\Omega_T} (s\rho)^{\tau_2+3}\xi e^{-2s\eta} vL^{-1}fdxdt - \iint_{\Omega_T} (s\rho)^{\tau_2+3}\xi e^{-2s\eta} vpq(0,x',t)h_{11}^1\partial_1 u(0,x',t)dxdt. \end{aligned}$$

We estimate each term of the right hand side of the previous equality. For example for the first one, using the definition of $I(\tau_2,v)$, Cauchy-Schwartz and Young inequalities, we can write :

$$\lambda^4 \iint_{\Omega_T} (s\rho)^{\tau_2+3}\xi e^{-2s\eta} v\partial_t K(\tilde{p_1},\tilde{q_1})udxdt = \lambda^4 \iint_{\Omega_T} \partial_t \left((s\rho)^{\tau_2+3}\xi e^{-2s\eta} v \right) K(\tilde{p_1},\tilde{q_1})udxdt,$$

then

$$\left|\lambda^4 \iint_{\Omega_T} \partial_t \left((s\rho)^{\tau_2+3}\xi e^{-2s\eta} v\right) K(\tilde{p_1},\tilde{q_1}) u dxdt\right| \le \left|\lambda^4 \iint_{\Omega_T} \partial_t \left((s\rho)^{\tau_2+3}\xi e^{-2s\eta}\right) v K(\tilde{p_1},\tilde{q_1}) u dxdt\right|$$
$$+\left|\lambda^4 \iint_{\Omega_T} (s\rho)^{\tau_2+3}\xi e^{-2s\eta}\partial_t(v) K(\tilde{p_1},\tilde{q_1}) u dxdt\right|$$
$$\le \varepsilon I(\tau_2,v) + \lambda^4 s^{\tau_2+4} \iint_{\Omega_T} (\rho)^{\tau_2+5}\xi e^{-2s\eta}(K(\tilde{p_1},\tilde{q_1})u)^2 dxdt$$
$$+\lambda^8 s^{\tau_2+7} \iint_{\Omega_T} (\rho)^{\tau_2+7}\xi e^{-2s\eta}(K(\tilde{p_1},\tilde{q_1})u)^2 dxdt$$
$$\le C_\varepsilon \lambda^8 s^{\tau_2+7} \|p\|_\infty^2 \|q\|_\infty^2 \int_0^T \int_{\omega'} \xi e^{-2s\eta}(\rho)^{\tau_2+7} \left(\int_0^{x_1} |u|^2 (y_1,x',t) dy_1\right) dx'dt + \varepsilon I(\tau_2,v)$$
$$\le C_\varepsilon \lambda^8 s^{\tau_2+7} \int_0^T \int_{\omega'} \left[\int_0^{\varepsilon} |u|^2(y_1,x',t) \left(\int_0^1 e^{-2s\eta}(\rho)^{\tau_2+7} dx_1\right) dy_1\right] dx'dt + \varepsilon I(\tau_2,v)$$
$$\le \varepsilon I(\tau_2,v) + C_\varepsilon \lambda^8 s^{\tau_2+7} \iint_{\omega_T} M(x',t)\,|u|^2\, dxdt$$

with $M(x',t) = \int_0^1 \rho^{\tau_2+7} e^{-2s\eta} dx_1$.
After technical calculations, we keep the higher exponents for s and λ, we obtain:

$$\lambda^4 \iint_{\widetilde{\omega}_T} (s\rho)^{\tau_2+3} e^{-2s\eta} |v|^2 dxdt \le \varepsilon I(\tau_2,v)$$
$$+C_\varepsilon \Bigg(\lambda^8 s^{\tau_2+7} \iint_{\omega_T} M(x',t)|u|^2 dxdt + \lambda^4 \iint_{\omega_T} e^{-2s\eta}(s\rho)^{3+\tau_2}|L^{-1}f|^2 dxdt$$
$$+\lambda^4 \int_0^T \int_{\omega'} \int_0^1 (s\rho)^{\tau_2+3} e^{-2s\eta} |\partial_1 u(0,x',t)|^2 dx_1 dx' dt \Bigg). \tag{2.19}$$

- Step 3: **Estimate of the boundary term**
Observe that for any f and h in $H^2(\omega)$,

$$\int_\omega h f \partial_1 f dx = -\frac{1}{2}\int_\omega |f|^2 \partial_1(h) dx + \frac{1}{2}\int_{\omega'} [(|f|^2 h)(\epsilon) - (|f|^2 h)(0)] dx'. \tag{2.20}$$

We denote $N(x',t) = \int_0^1 \rho^{\tau_2+3} e^{-2s\eta} dx_1$, $n^* = 2\tau_2 - \tau_1 + 7$. We have

$$\begin{aligned}\lambda^4 \iint_{\omega_T} (s\rho)^{\tau_2+3} e^{-2s\eta} |\partial_1 u(0,x',t)|^2 dxdt &= \lambda^4 s^{\tau_2+3} \int_0^T \int_{\omega'} \left(\int_0^1 (\rho)^{\tau_2+3} e^{-2s\eta} dx_1\right) |\partial_1 u(0,x',t)|^2 dx'dt \\ &= \lambda^4 s^{\tau_2+3} \int_0^T \int_{\omega'} N(x',t) |\partial_1 u(0,x',t)|^2 dx'dt.\end{aligned}$$

We apply (2.20) with $f = \partial_1 u$, and $h = N\xi$ and obtain

$$\lambda^4 \iint_{\omega_T} (s\rho)^{\tau_2+3} e^{-2s\eta} |\partial_1 u(0,x',t)|^2 dxdt =$$
$$2\lambda^4 s^{\tau_2+3} \iint_{\Omega_T} \partial_1^2 u\, \partial_1 u\, \xi N(x',t) dxdt - \lambda^4 s^{\tau_2+3} \int_{\Omega_T} |\partial_1 u|^2 N(x',t) \partial_1 \xi dxdt := I_1 - I_2.$$

From Young inequality, we deduce that

$$|I_1| \le \varepsilon \iint_{\Omega_T} (s\rho)^{\tau_1-1} e^{-2s\eta^*} |\partial_1^2 u|^2 dxdt + \frac{1}{4\varepsilon} \lambda^8 s^{n^*} \iint_{\omega_T} \rho^{-\tau_1+1} e^{2s\eta^*} \xi^2 N^2(x',t) |\partial_1 u|^2 dxdt.$$

The first term in the right hand side is estimated as follows

$$\iint_{\Omega_T} (s\rho)^{\tau_1-1}e^{-2s\eta^*}|\partial_1^2 u|^2dxdt \le I(\tau_1,u).$$

For the second term, integrating by parts and using that $(\xi\, u)$ vanishes at the boundary of ω, we get

$$\begin{aligned}\lambda^8\iint_{\omega_T}\rho^{-\tau_1+1}s^{n^*}e^{2s\eta^*}\,\xi^2N^2(x',t)|\partial_1 u|^2dxdt &= -\lambda^8\iint_{\omega_T}\rho^{-\tau_1+1}s^{n^*}e^{2s\eta^*}\,\xi^2N^2(x',t)u\,\partial_1^2u\,dxdt\\ &-\ \lambda^8\iint_{\omega_T}\partial_1\left(\rho^{-\tau_1+1}s^{n^*}e^{2s\eta^*}\xi^2\,N^2(x',t)\right)u\,\partial_1 udxdt\\ &:=\ P_1+P_2\end{aligned}$$

As before, using Young estimate, for the first term P_1, we have

$$|P_1| \le \varepsilon^2 I(\tau_1,u)+\lambda^{16}\frac{1}{4\varepsilon^2}\iint_{\omega_T}\rho^{-3\tau_1+3}s^{4\tau_2-3\tau_1+15}e^{6s\eta^*}N^4(x',t)|u|^2dxdt.$$

Noting that

$$N(x,t)\ \le e^{-2sn_-}(\rho^*)^{\tau_2+3},\quad \forall (x',t)\in\omega'_T,$$

we deduce that

$$|P_1| \le \varepsilon^2 I(\tau_1,u)+\lambda^{16}C_\varepsilon\iint_{\omega_T}(s\rho^*)^{4\tau_2-3\tau_1+15}e^{-8s\eta_-+6s\eta^*}|u|^2dxdt,$$

where C_ε denotes a positive constant depending en ε. To estimate P_2 we use (2.20) with $f=u$ and $h=\partial_1(\rho^{-\tau_1+1}s^{n^*}e^{2s\eta^*}\,\xi^2N^2(x',t))$. Taking into account that fh vanishes on the boundary of ω, we obtain

$$P_2=\frac{\lambda^8}{2}\iint_{\omega_T}\partial_{x_1}^2\left(\rho^{-\tau_1+1}s^{n^*}e^{2s\eta^*}\xi^2\,N^2(x',t)\right)|u|^2\,dxdt.$$

It remains to estimate I_2 of (2.21):

$$I_2=\lambda^4s^{\tau_2+3}\int_{\Omega_T}\partial_1 u\,\partial_1 u\,N(x',t)\partial_1\xi dxdt.$$

Integrating by parts, and using that $\partial_1\xi\, u$ vanishes on the boundary of ω, we get

$$I_2=-\lambda^4s^{\tau_2+3}\int_{\Omega_T}u\,\partial_1^2u\,N(x',t)\partial_1\xi dxdt-\lambda^4s^{\tau_2+3}\int_{\Omega_T}u\partial_1 u\,N(x',t)\partial_1^2\xi dxdt:=J_1+J_2.$$

As previously,

$$|J_1| \le \varepsilon I(\tau_1,u)+\lambda^8C_\varepsilon\ s^{n^*}\iint_{\omega_T}\rho^{-\tau_1+1}e^{2s\eta^*}(\partial_1\xi)^2N^2(x',t)|u|^2dxdt.$$

Integrating by parts J_2 and using that $u\partial_{x_1}^2\xi$ vanishes on the boundary, we obtain:

$$J_2=\frac{\lambda^4s^{\tau_2+3}}{2}\int_{\Omega_T}|u|^2\,N(x',t)\partial_1^3\xi dxdt.$$

Summarizing all the previous estimates, we get that for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\begin{aligned}\lambda^4 \int_0^T \int_{\omega'} \int_0^1 (s\rho)^{\tau_2+3} e^{-2s\eta} |\partial_1 u(0,x',t)|^2 dx_1 dx' dt \leq \varepsilon I(\tau_1, u) \\ +\lambda^{16} C_\varepsilon \iint_{\omega_T} (s\rho^*)^{4\tau_2-3\tau_1+15} e^{-8s\eta_- + 6s\eta^*} |u|^2 dxdt \\ +\frac{\lambda^8}{2} \iint_{\omega_T} |\partial_{x_1}^2 \left(\rho^{-\tau_1+1} s^{n^*} e^{2s\eta^*} \xi^2\, N^2(x',t)\right)|\, |u|^2 dxdt \\ +\lambda^8 C_\varepsilon\, s^{n^*} \iint_{\omega_T} \rho^{-\tau_1+1} e^{2s\eta^*} (\partial_1 \xi)^2 N^2(x',t) |u|^2 dxdt \\ +\frac{\lambda^4 s^{\tau_2+3}}{2} \int_{\Omega_T} |u|^2\, N(x',t) |\partial_1^3 \xi| dxdt.\end{aligned}$$

It remains to verify that the weights in the last 4 integrals are bounded. The weight in the last integral is clearly bounded. Moreover, one has the following lemma:

LEMMA 2.10. *Let $a \in \mathbb{R}$. There exist $\lambda_0 \in \mathbb{R}$, $s_0 = s(\lambda_0)$ such that*

$$\rho^a\, e^{-s\eta} \leq 1 \quad \forall \lambda \geq \lambda_0,\ s \geq s_0. \tag{2.21}$$

See Appendix 6.2 for a proof.

Applying Lemma 2.10 with $a = \tau_2 + 3$, we deduce that $N^2(x',t) \leq e^{-3s\eta_-}$. So

$$\lambda^8 C_\varepsilon \iint_{\omega_T} s^{n^*} \rho^{-\tau_1+1} e^{2s\eta^*} (\partial_1 \xi)^2 N^2(x',t) |u|^2 dxd \leq \lambda^8 C_\varepsilon \iint_{\omega_T} s^{n^*} \rho^{-\tau_1+1} e^{s(2\eta^* - 3\eta_-)} |u|^2 dxdt$$

where C_ε denotes any constant depending on ε and independent on s, λ. Moreover,

$$\partial_1^2 \left(\rho^{-\tau_1+1} s^{n^*} e^{2s\eta^*} \xi^2\, N^2(x',t)\right) = s^{n^*} N^2(x',t) e^{2s\eta^*} \partial_1^2 \left(\rho^{-\tau_1+1} \xi^2\right),$$

and

$$|\partial_1^2 \xi^2 \rho^{-\tau_1+1}| \leq C \lambda^2 \rho^{-\tau_1+1},$$

with C a constant depending on β and ξ. So

$$|\partial_1^2 \left(\rho^{-\tau_1+1} s^{n^*} e^{2s\eta^*} \xi^2\, N^2(x',t)\right)| \leq C s^{n^*} \lambda^2 \rho^{-\tau_1+1} e^{s(2\eta^* - 3\eta_-)}$$

Choosing $\lambda_0 \geq 1$,the previous computations leads to:

$$\begin{aligned}\lambda^4 \int_0^T \int_{\omega'} \int_0^1 (s\rho)^{\tau_2+3} e^{-2s\eta} |\partial_1 u(0,x',t)|^2 dx_1 dx' dt \leq \varepsilon I(\tau_1, u) \\ +\lambda^{16} C_\varepsilon \iint_{\omega_T} (s\rho^*)^{\tau^*} e^{-8s\eta_- + 6s\eta^*} |u|^2 dxdt,\end{aligned}$$

where $\tau^* := \max\{n^*, 1 - \tau_1, 4\tau_2 - 3\tau_1 + 15, \tau_2 + 3\}$. The last step is to verify that one can choose η such that

$$-4\eta_- + 3\eta^* < 0. \tag{2.22}$$

Recalling that

$$\eta(x,t) = \frac{e^{2\lambda K} - e^{\lambda\beta(x)}}{t(T-t)},$$

we obtain that assumption (2.22) is checked for

$$K \geq \max\{\frac{2\ln 2}{\|\beta\|_\infty}, \|\beta\|_\infty\}. \tag{2.23}$$

Returning to (2.19) and noting that $N \leq M$, we obtain:

$$\begin{aligned}\lambda^4 \iint_{\widetilde{\omega_T}} (s\rho)^{\tau_2+3} e^{-2s\eta}|v|^2 dxdt \leq{}& \varepsilon I(\tau_1, u) + \varepsilon I(\tau_2, v)\\ &+C_\varepsilon \lambda^{16} \iint_{\omega_T} (s\rho)^{\tau^*} e^{-2s\alpha}|u|^2 dxdt\\ &+C_\varepsilon\ \lambda^8\ s^{\tau_2+7} \iint_{\omega_T} M(x',t)|u|^2 dxdt,\\ &+\lambda^4 \iint_{\omega_T} e^{-2s\eta}(s\rho)^{3+\tau_2}|L^{-1}f|^2 dxdt.\end{aligned}$$

The proof is ended for the case where $B = (1, 0, ..., 0)$ by noting that

$$\lambda^8 s^{\tau_2+7} \iint_{\omega_T} M(x',t)|u|^2 dxdt \leq \lambda^{16} \iint_{\omega_T} (s\rho)^{\tau^*}\ e^{-2s\alpha}|u|^2 dxdt.$$

The proof in the general case is done transforming the general system to this simplest one using the result in the Appendix 6.1 □

Finally using Theorem 2.8 in (2.13), we obtain Theorem 2.6. □

3. Carleman Estimate for 3×3 Systems.

3.1. Statement of the problem. In this section we prove the main result of this paper, i.e., Theorem 1.1, that is the null controllability under appropriate conditions of the 3×3 reaction-diffusion system

$$\begin{cases} \partial_t y = (\mathcal{L} + A)y + Cf1_\omega \text{ in } \Omega_T, \\ y = 0 \text{ on } \Sigma_T, \\ y(\cdot, 0) = y_0(\cdot) \text{ in } \Omega, \end{cases} \tag{3.1}$$

where,

$$\mathcal{L} = \begin{pmatrix} div(H_1\nabla) & 0 & 0 \\ 0 & div(H_2\nabla) & 0 \\ 0 & 0 & div(H_2\nabla) \end{pmatrix},$$

$A = (a_{ij})_{1\leq i,j\leq 3}$, $C = (1,0,0)^t \in \mathcal{L}\left(\mathbb{R}, \mathbb{R}^3\right)$, $f \in L^2(\Omega_T)$ and $y_0 = (y_{0,i})_{1\leq i\leq 3} \in L^2(\Omega)^3$. Note that in this system we are taking $H_2 = H_3$. The case in which all the H_i are different is an open problem and is related with a controllability problem of two parabolic equations with a second order coupling (see [21]). As in the two dimensional case the result is reduced to proving an observability inequality to the adjoint system to (3.1). That is, consider

$$\begin{cases} -\partial_t\varphi_1 = div(H_1\nabla\varphi_1) + a_{11}\varphi_1 + a_{21}\varphi_2 + a_{31}\varphi_3 & \text{in} \quad \Omega_T, \\ -\partial_t\varphi_2 = div(H_2\nabla\varphi_2) + a_{12}\varphi_1 + a_{22}\varphi_2 + a_{32}\varphi_3 & \text{in} \quad \Omega_T, \\ -\partial_t\varphi_3 = div(H_2\nabla\varphi_3) + a_{13}\varphi_1 + a_{23}\varphi_2 + a_{33}\varphi_3 & \text{in} \quad \Omega, \\ \varphi_1 = \varphi_2 = \varphi_3 = 0 & \text{on} \quad \Sigma_T, \\ \varphi_1(\cdot,T) = \varphi_1^T,\, \varphi_2(\cdot,T) = \varphi_2^T,\, \varphi_3(\cdot,T) = \varphi_3^T, & \text{in } \Omega \end{cases}$$

Then, Theorem 1.1 is equivalent to the following result:

THEOREM 3.1. *Suppose that $a_{ij} \in C^4(\overline{\Omega_T})$, and that there exists $j \in \{2,3\}$ such that $|a_{j1}(x,t)| \geq C > 0$ for all $(x,t) \in \omega_T$. For such j we define $k_j = \frac{6}{j}$,*

$$B_{k_j} := -2H_2\left(\nabla a_{k_j1} - \frac{a_{k_j1}}{a_{j1}}\nabla a_{j1}\right),$$

$$b_j = \frac{2H_2\nabla a_{j1}(\nabla a_{k_j1}a_{j1} - \nabla a_{j1}a_{k_j1})}{a_{j1}^2} + \frac{a_{k_j1}div(H_2\nabla a_{j1}) - a_{j1}div(H_2\nabla a_{k_j1})}{a_{j1}}$$

$$+\frac{a_{j1}\partial_t a_{k1} - a_{k1}\partial_t a_{j1}}{a_{j1}} - (-1)^j\frac{\det\left[A\,|\,C\right]}{a_{j1}}.$$

Assume that $\partial\omega \cap \partial\Omega = \gamma$, $|\gamma| \neq 0$, and $B_{k_j}\cdot\nu \neq 0$, on γ_T. Then, there exists $C > 0$ such that for every $(\varphi_1^T, \varphi_2^T, \varphi_3^T) \in L^2(\Omega)^3$ the corresponding solution to (3.1) satisfies:

$$\int_\Omega (|\varphi_1(x,0)|^2 + |\varphi_2(x,0)|^2 + |\varphi_3(x,0)|^2)dx \leq C\iint_{\omega_T}|\varphi_1(x,t)|^2dxdt. \tag{3.2}$$

Observe that the case $B_{k_j}(x,t) = 0$ and $b_j(x,t) \neq 0$ was already treated in [3]. The really new result is the case $B_{k_j}\cdot\nu \neq 0$, on γ_T. Inequality (3.2) will be deduced by an appropriate Carleman estimate (as in the two dimensional case treated below). So, the next subsection is devoted to the proof of this Carleman inequality.

3.2. A new Carleman estimate for a 3×3 reaction-diffusion system with one observation. In view of applications regarding inverse problems we will consider the following 3×3 reaction-diffusion system (which is the adjoint system of system (3.1)) where $(f,g,h) \in L^2(\Omega)^3$.

Let $(a_{ij})_{1\leq i,j\leq 3} \in C^4(\overline{\Omega_T})$, $H_l = (h_{ij}^l)_{1\leq i,j\leq 3}$, $1 \leq l \leq 2$ defined by (1.2) and (1.3). We consider the following system:

$$\begin{cases} \partial_t u = div(H_1\nabla u) + a_{11}u + a_{21}v + a_{31}w + f & \text{in} \quad \Omega_T, \\ \partial_t v = div(H_2\nabla v) + a_{12}u + a_{22}v + a_{32}w + g & \text{in} \quad \Omega_T, \\ \partial_t w = div(H_2\nabla w) + a_{13}u + a_{23}v + a_{33}w + h & \text{in} \quad \Omega, \\ u = v = w = 0 & \text{on} \quad \Sigma_T, \\ u(\cdot,0) = u_0,\, v(\cdot,0) = v_0,\, w(\cdot,0) = w_0, & \text{in } \Omega. \end{cases} \tag{3.3}$$

Uniqueness, existence and stability results for (3.3) can be proved by classical theory (e.g., [26]). In particular, (3.3) admits an unique solution $(u,v,w) \in C([0,T];L^2(\Omega))^3 \cap L^2(0,T;H_0^1(\Omega))^3$.

Moreover, if $u_0, v_0, w_0 \in H^2(\Omega)\cap H_0^1(\Omega)$, then $(u,v,w) \in C([0,T]; H^2(\Omega)\cap H_0^1(\Omega))^3 \cap C^1([0,T]; L^2(\Omega))^3$ and we will call it *strong solution*.

The Carleman estimate (2.10) allows us to perform a new Carleman estimate for this 3×3 reaction-diffusion-system. First of all, let us precise the assumptions on the coefficients of system (3.3).

ASSUMPTION 3.2.

1. *$\omega \subset \Omega$ is a non-empty subdomain of class C^2 with $\partial\omega \cap \partial\Omega = \gamma$ and $|\gamma| \neq 0$,*
2. *There exists $j \in \{2,3\}$ such that $|a_{j1}(x,t)| \geq C > 0$ for all $(x,t) \in \omega_T$ and for $k_j = \dfrac{6}{j}$*
$$H_2\left(\nabla a_{k_j 1} - \frac{a_{k_j 1}}{a_{j1}}\nabla a_{j1}\right)\cdot \nu \neq 0, \quad on\ \gamma_T,$$
3. *Let $H_2|_{\omega_T} \in (W^{3,\infty}(\omega_T))^{n^2}$.*

REMARK 3.3. *In this result, we are again assuming that $B_{k_j} \neq 0$ in ω_T. The Carleman inequality is still true in the other case and it was already proved in [3].*
We obtain the following theorem :

THEOREM 3.4. *Under Assumption 3.2 there exist a positive function $\beta \in C^2(\overline{\Omega})$ (only depending on Ω and ω) and three positive constants s_0, λ_0, C (which only depend on Ω, ω, $(a_{ij})_{(1\leq i,j\leq 3)}$, τ), a constant K (see (3.7)) such that for every $(u_0,v_0,w_0) \in L^2(\Omega)^3, (f,g,h) \in L^2(\Omega_T)^3$, the following Carleman estimate holds*

$$\begin{aligned} I(\tau,u)+I(\tau,v)+I(\tau,w) \leq C\Bigg(\lambda^{32}\iint_{\omega_T} & s^{(\tau+33)}(\rho^*)^{\tau+31}e^{(-4s\alpha+2s\eta)}(|u|^2+|f|^2)dxdt \\ +\lambda^4 \iint_{\omega_T} & e^{-2s\eta}(s\rho)^{3+\tau}(|Qg|^2+|Qh|^2)dxdt \\ + \iint_{\Omega_T} & e^{-2s\eta}(s\rho)^{\tau}(|f|^2+|g|^2+|h|^2)dxdt\Bigg). \end{aligned} \tag{3.4}$$

for $s \geq s_0$, $\lambda \geq \lambda_0$ and all (u,v,w) solution of (3.3), η defined by (2.6), and Q defined in (6.5), $\alpha = 4\eta_- - 3\eta^*$, $\eta^* = \max_{\overline{\Omega}} \eta$, $\eta_- = \min_{\overline{\Omega}} \eta$ and $\rho^* = \max_{\overline{\Omega}} \rho$.

Proof. [Proof of Theorem 3.4] The proof is done in three steps:

1. We first prove a Carleman estimate with three observations taking sets $\widetilde{\widetilde{\omega}} \subset\subset \widetilde{\omega} \subset \omega$, such that $\widetilde{\gamma} = \partial\widetilde{\omega}\cap\partial\Omega \subset \gamma$ and dist $(\partial\widetilde{\omega}\backslash\widetilde{\gamma}, \partial\omega\backslash\gamma) > 0$ (this is necessary technicality that allows to construct ξ satisfying (2.18). A direct application of Theorem 2.4 leads to
$$\begin{aligned} I(\tau,u)+I(\tau,v)+I(\tau,w) \leq C\Bigg(\lambda^4 \iint_{\widetilde{\widetilde{\omega}}_T} (s\rho)^{\tau+3}e^{-2s\eta}|u|^2dxdt \\ +\lambda^4 \iint_{\widetilde{\widetilde{\omega}}_T} (s\rho)^{\tau+3}e^{-2s\eta}|v|^2dxdt \quad +\lambda^4 \iint_{\widetilde{\widetilde{\omega}}_T} (s\rho)^{\tau+3}e^{-2s\eta}|w|^2dxdt \\ +\iint_{\Omega_T}(s\rho)^{\tau}e^{-2s\eta}|f|^2dxdt + \iint_{\Omega_T}(s\rho)^{\tau}e^{-2s\eta}|g|^2dxdt + \iint_{\Omega_T}(s\rho)^{\tau}e^{-2s\eta}|h|^2dxdt\Bigg), \end{aligned}$$
for any $\widetilde{\widetilde{\omega}} \subset\subset \Omega$. The main question is to get rid of
$$\iint_{\widetilde{\widetilde{\omega}}_T} (s\rho)^{\tau+3}e^{-2s\eta}|v|^2dxdt + \iint_{\widetilde{\widetilde{\omega}}_T} (s\rho)^{\tau+3}e^{-2s\eta}|w|^2dxdt.$$

2. We eliminate two observations. Let

$$z = a_{21}v + a_{31}w \quad \text{in } \omega_T. \tag{3.5}$$

Suppose, for example, that Assumption 3.2 is satisfied for $j = 3$. If (u, v, w) is a strong solution of system (3.3), then z defined by (3.5) satisfies:

$$\text{(3.6)}\left\{\begin{array}{lll} \partial_t u = div(H_1\nabla u) + a_{11}u + z + f & \text{in} & \omega_T, \\ \partial_t z = div(H_2\nabla z) + A\cdot\nabla z + az + eu + B\cdot\nabla v + bv + G & \text{in} & \omega_T, \end{array}\right.$$

with

$$b = 2H_2\nabla a_{31}\left(\frac{\nabla a_{21}a_{31} - \nabla a_{31}a_{21}}{a_{31}^2}\right) + \frac{a_{21}div(H_2\nabla a_{31}) - a_{31}div(H_2\nabla a_{21})}{a_{31}}$$
$$+\frac{a_{31}\partial_t a_{21} - a_{21}\partial_t a_{31}}{a_{31}} - \frac{a_{21}^2 a_{32} + a_{31}a_{21}a_{33} - a_{31}a_{21}a_{22} - a_{31}^2 a_{23}}{a_{31}},$$

$$A = -2H_2\frac{\nabla a_{31}}{a_{31}}, \quad B = -2H_2\left(\nabla a_{21} - \frac{a_{21}}{a_{31}}\nabla a_{31}\right),$$

$$a = 2\frac{(H_2\nabla a_{31})\cdot\nabla a_{31}}{a_{31}^2} - \frac{div(H_2\nabla a_{31})}{a_{31}} + \frac{a_{21}a_{32} + a_{31}a_{33}}{a_{31}} + \frac{\partial_t a_{31}}{a_{31}},$$

$$e = a_{12}a_{21} + a_{13}a_{31}, \quad G = a_{21}g + a_{31}h.$$

We first use Theorem 2.8 to estimate v by z, u and G. Then with Assumption 3.2 and (3.5), we have $w = \dfrac{z - a_{21}v}{a_{31}}$. Therefore v and w will be estimated by z (and u, G). It will remain to estimate z by u. Of course, all these estimates are locally (in ω_T). We apply Theorem 2.8 to the second equation of (3.6), we obtain:

$$\lambda^4 \iint_{\widetilde{\omega}_T} (s\rho)^{\tau+3}e^{-2s\eta}|v|^2 dxdt \leq C_\varepsilon\ \lambda^{16} \iint_{\widetilde{\omega}_T} e^{-2s\alpha}\ (s\rho^*)^{\tau+15}|z|^2 dxdt$$
$$+C\lambda^4 \iint_{\omega_T} e^{-2s\eta}(s\rho)^{3+\tau}(|Qu|^2 + |Qg|^2 + |Qh|^2)dxdt$$
$$+\varepsilon I(\tau, v).$$

Now we are going to estimate the local observation in z by the local observation in u using the first equation of (3.6). For this, we multiply this equation by $\lambda^{16}(s\rho^*)^{\tau+15}\xi e^{-2s\alpha}z$ where ξ is defined in (2.18) and we integrate on ω_T.

$$\lambda^{16} \iint_{\widetilde{\omega}_T} (s\rho^*)^{\tau+15}e^{-2s\alpha}|z|^2 dxdt$$

$$\leq \lambda^{16}\left(\iint_{\omega_T} (s\rho^*)^{\tau+15}e^{-2s\alpha}z(x,t)\xi(\partial_t u - div(H_1\nabla u) - a_{11}u - f)dxdt\right).$$

Using integrations by part we obtain

$$\lambda^{16} \iint_{\widetilde{\omega}_T} e^{-2s\alpha}\ (s\rho^*)^{\tau+15}\ |z|^2 dxdt \leq \epsilon I(\tau, z)$$
$$+C_\epsilon\lambda^{32}\iint_{\omega_T} s^{(\tau+33)}(\rho^*)^{(\tau+31)}e^{(-4s\alpha+2s\eta)}(|u|^2 + |f|^2)dxdt.$$

The integral in the right hand side of the previous inequality is bounded if:

$$8\eta_- - 7\eta^* > 0.$$

In others terms we need

$$K \geq \max\{\frac{3\ln 2}{\|\beta\|_\infty}, \|\beta\|_\infty\}. \tag{3.7}$$

Finally we obtain :

$$\begin{aligned} I(\tau,u)+I(\tau,v)+I(\tau,w) \leq C\Bigg(&\lambda^{32}\iint_{\omega_T} s^{(\tau+33)}(\rho^*)^{(\tau+31)}e^{(-4s\alpha(t)+2s\eta)}(|u|^2+|f|^2)dxdt \\ &+\lambda^4\iint_{\omega_T} e^{-2s\eta}(s\rho)^{3+\tau}(|Qg|^2+|Qh|^2)dxdt \\ &+\iint_{\Omega_T} e^{-2s\eta}(s\rho)^{\tau}(|f|^2+|g|^2+|h|^2)dxdt\Bigg). \end{aligned}$$

□

REMARK 3.5. *Following the proofs presented here it is clear that the regularity assumptions on* a_{jk} *and* A_{jk} *are not optimal. For simplicity, we asked for much more regularity than the necessary. In fact, in Theorem 1.1 we can assume that* $a_{jk} \in L^\infty(\Omega_T)$ *for every* $j,k \in \{1,2,3\}$*,* $a_{kj}|_{\omega_T} \in W^{2,\infty}(\omega_T)$ *for every* $j,k \in \{2,3\}$ *and* $a_{k,1}|_{\omega_T} \in W^{4,\infty}(\omega_T)$ *for* $k \in \{2,3\}$*.*

4. Applications and Generalizations.

4.1. Controllability to trajectories of class of 3×3 nonlinear parabolic systems.

THEOREM 4.1. *Let* $a_{kj}|_{\omega_T} \in W^{2,\infty}(\omega_T)$ *for every* $j,k \in \{2,3\}$ *and* $a_{k,1}|_{\omega_T} \in W^{4,\infty}(\omega_T)$ *for* $k \in \{2,3\}$*,* a_{21}, a_{31} *and* ω *satisfy Assumption 3.2. Suppose that* $F : \mathbb{R}^3 \to \mathbb{R}$*, is a locally Lipschitz function with respect to each variable. Let*

$$\left\{\begin{array}{lll} \partial_t y_1 = div(H_1\nabla y_1) + F(y_1,y_2,y_3) + \chi_\omega f & in & \Omega_T, \\ \partial_t y_2 = div(H_2\nabla y_2) + a_{21}y_1 + a_{22}y_2 + a_{23}y_3 & in & \Omega_T, \\ \partial_t y_3 = div(H_2\nabla y_3) + a_{31}y_1 + a_{32}y_2 + a_{33}y_3 & in & \Omega, \\ y_1 = y_2 = y_3 = 0 & on & \Sigma_T, \\ y_1(\cdot,0) = y_1^0,\ y_2(\cdot,0) = y_2^0,\ y_3(\cdot,0) = y_3^0, & in\ \Omega. & \end{array}\right. \tag{4.1}$$

Then there is $\rho > 0$ *such that if* $\left\|y_i^0\right\|_{L^\infty(\Omega)} \leq \rho$ *for all* $1 \leq i \leq 3$*, one can find* $f \in L^2(Q_T)$ *such that there exists* (y_1,y_2,y_3) *solution of (4.1) satisfying:*

$$y_1(\cdot,T) = y_2(\cdot,T) = y_3(\cdot,T) = 0 \quad in\ \Omega.$$

Proof. The proof is by now classical. It is based on an observability estimate for the linearized system. A inspection of the proof of Theorem 3.4 shows that we only used the derivatives of the terms a_{kj} for $k \neq 1$. So the Carleman inequality for the adjoint system to (3.1) remains valid when no other regularity assumptions are done in a_{1k} except that they belong to $L^\infty(\Omega_T)$. Then, the local exact controllability to trajectories of system 4.1 follows from (3.4) and the Kakutani's fixed point theorem (see for example [15], [3], [19]).□

4.2. Generalization for $n \times n$ reaction diffusion systems for $n \geq 3$. The results of Section 3 can be generalized to $n\times n$ parabolic systems controlled by $(n-2)$ controls. Consider $A = (a_{lm})_{1\leq l,m\leq n}$ a matrix of order n with $a_{lm} \in C^4(\overline{\Omega_T})$. For $j \neq k \in \{1,...,n\}$ consider $C_{jk} = (\rho_1, \rho_2, ..., \rho_n)^t$ with $\rho_l = e_l$ (where $(e_1,...,e_n)$ is the euclidian basis of $\mathbb{R}^n$) except for the two indexes j,k where $\rho_j = \rho_k = 0$. We assume that j,k can be chosen in such a way that there exists $i \neq j$, $i \neq k$ such that $|a_{ji}(x,t)| \geq C > 0$ for all $(x,t) \in \omega_T$. We denote $B := -2\left(\nabla a_{ki} - \frac{a_{ki}}{a_{ji}}\nabla a_{ji}\right)$ and

$$b = 2\nabla a_{ji}\left(\frac{\nabla a_{ki}a_{ji} - \nabla a_{ji}a_{ki}}{a_{ji}^2}\right) + \frac{a_{ki}\Delta a_{ji} - a_{ji}\Delta a_{ki}}{a_{ji}}$$
$$+\frac{a_{ji}\partial_t a_{ki} - a_{ki}\partial_t a_{ji}}{a_{ji}} - \frac{a_{ki}^2 a_{ji} + a_{ji}a_{ki}a_{jj} - a_{ji}a_{ki1}a_{kk} - a_{ji}^2 a_{kij}}{a_{ji}}.$$

Furthermore, we assume that all the following conditions are checked

1. $a_{m,k} = 0$ for $m \neq k, j, i$.
2. $a_{m,j} = 0$ for $m \neq k, j, i$.
3. $\partial\omega \cap \partial\Omega = \gamma$, $|\gamma| \neq 0$,
4. $B\cdot\nu \neq 0$, on γ_T.

THEOREM 4.2. *Under the previous assumptions, the following system in $L^2(\Omega\times (0,T))^n$.*

$$\left\{\begin{array}{lll} \partial_t Y = \Delta Y + AY + C_{jk}U\chi_\omega & in & \Omega_T, \\ Y(\cdot,t) = 0 & on & \Sigma_T, \\ Y(\cdot,0) = Y_0(\cdot) & in & \Omega. \end{array}\right. \tag{4.2}$$

is null controllable.

Proof. Controllability of system (4.2) is equivalent to the following observability estimate:

$$\exists C > 0; \|\Phi(T)\|^2_{L^2(\Omega)^n} \leq C\sum_{l=1,l\neq j,k}^{n}\iint_{\omega_T}|\phi_l(x,t)|^2 dxdt,$$

for all $\Phi = (\phi_1,...,\phi_n)^t$ solution of the adjoint system:

$$\left\{\begin{array}{lll} \partial_t\Phi = \Delta\Phi + A^*\Phi & \text{in} & \Omega_T, \\ \Phi(\cdot,t) = 0 & \text{on} & \Sigma_T. \end{array}\right. \tag{4.3}$$

By scalar Carleman estimate applied to each equation of system (4.3), we obtain

$$\sum_{l=1}^{l=n} I(\tau,\phi_l) \leq C\sum_{l=1}^{l=n}\iint_{\omega_T}(s\rho)^{\tau+3}e^{-2s\eta}|\phi_l(x,t)|^2 dxdt. \tag{4.4}$$

So (ϕ_i,ϕ_j,ϕ_k) satisfies

$$\left\{\begin{array}{lll} \partial_t\phi_i = \Delta\phi_i + \sum_{l=1}^n a_{li}\phi_l & \text{in} & \Omega_T, \\ \partial_t\phi_j = \Delta\phi_j + a_{ij}\phi_i + a_{jj}\phi_j + a_{kj}\phi_k & \text{in} & \Omega_T, \\ \partial_t\phi_k = \Delta\phi_k + a_{ik}\phi_i + a_{jk}\phi_j + a_{kk}\phi_k & \text{in} & \Omega_T, \\ \Phi(\cdot,t) = 0 & \text{on} & \Sigma_T, \\ \Phi(\cdot,0) = \Phi_0(\cdot) & \text{in} & \Omega, \end{array}\right.$$

satisfy the following Carleman estimate:

$$I(\tau,\phi_i)+I(\tau,\phi_j)+I(\tau,\phi_k)\leq C\left(\lambda^{32}\iint_{\omega_T} s^{(\tau+33)}(\rho^*)^{(\tau+31)}e^{(-4s\alpha(t)+2s\eta)}(|\phi_i|^2+|f|^2)dxdt\right.$$
$$\left.+\iint_{\Omega_T} e^{-2s\eta}(s\rho)^{\tau}|f|^2dxdt\right). \tag{4.5}$$

where $f=\sum_{l=1,l\neq i,j,k}^{n} a_{li}\phi_l$. It is now straightforward to see that a combination of (4.4) and (4.5) gives

$$\sum_{l=1}^{l=n} I(\tau,\phi_l)\leq C\sum_{l=1;l\neq k,j}^{l=n}\iint_{\omega_T}(s\rho)^{\tau+3}e^{-2s\eta}|\phi_l(x,t)|^2dxdt.$$

Classical energy estimates give now the observability inequality and therefore the null controllability result. □

REMARK 4.3. *Observe that the use of Carleman inequality (1.1) involves terms that include the operator Q defined in (6.5). This imposes the previous conditions 1 and 2 on the coefficients a_{mk} and a_{mj} for $m\neq i,j,k$.*

All the regularity assumptions of system (4.2) can be weakened at it has been pointed out in Remark 3.5.

4.3. Inverse Problems. This subsection is devoted to the question of the identification of coefficients for a reaction-diffusion system of n equations ($n\geq 3$) in a bounded domain, with the main particularity that we observe *only* $(n-2)$ *components of the system*. Such reconstruction result can be applied to several models involving 3×3 linear reaction-diffusion system. For example, it is a classical approach to treat Prey-Predator models with prey-stage structure (see [30] and references therein).
In the literature, very few results exist concerning the reconstruction of coefficients in parabolic systems. We can cite [31] in which the authors obtain the identification of the principal part of a parabolic equation using a large set of inputs. In a previous result [8], the authors obtain the reconstruction of all the coefficients in a 2×2 reaction-diffusion-advection system by repeated measurements of one component. In the following result our goal is to minimize the number of components involved for the reconstruction of the coefficients in a $n\times n$ reaction-diffusion system. We claim that the identification of n coefficients (one in each equation) via a Lipschitz inequality is possible basing ourselves on the local measurement of only $n-2$ components. However, we are able to reconstruct all the coefficients using well suited repeated measurements of one (resp. $n-2$) component(s) in a 3×3 (resp. $n\times n$) linear parabolic system. The main idea used for this task is similar as one used in [23] or in [8]. Furthermore, the Carleman estimate obtained in [8] for a 3×3 reaction-diffusion system differs from one obtained in this paper by the norm used: $W_2^{2,1}(\omega_T)$ is the norm used in the first one and a L^2 norm in the second one (see Remark 2.7). In view of numerical implementations, Theorem 4.4 allows us to proceed with less computations and the convergence of a scheme based on a L^2 norm is easier to obtain than for a stronger norm. The key ingredient is the stability result for 3×3 reaction diffusion systems and the case of n equations is a direct consequence, so let us focus for $n=3$.

For simplicity in the exposition of the following result we will consider the case in which all the coefficients in the system are time independent. In fact this condition is only used for the coefficients we want to recover.

Consider the following 3×3 reaction-diffusion system where $(a_{ij})_{1\leq i,j\leq 3} \in C^4(\overline{\Omega})^9$:

$$\begin{cases} \partial_t U = \Delta U + a_{11}U + a_{21}V + a_{31}W & \text{in} \quad \Omega_T, \\ \partial_t V = \Delta V + a_{12}U + a_{22}V + a_{32}W & \text{in} \quad \Omega_T, \\ \partial_t W = \Delta W + a_{13}U + a_{23}V + a_{33}W & \text{in} \quad \Omega_T, \\ U = k_1,\ V = k_2,\ W = k_3 & \text{on} \quad \Sigma_T, \\ U(.,0) = U_0 \quad V(.,0) = V_0 \quad \text{and} \quad W(.,0) = W_0 & \text{in} \quad \Omega. \end{cases}$$

Let $(\widetilde{U}, \widetilde{V}, \widetilde{W})$ be solution of

$$\begin{cases} \partial_t \widetilde{U} = \Delta \widetilde{U} + a_{11}\widetilde{U} + \tilde{a}_{21}\widetilde{V} + a_{31}\widetilde{W} & \text{in} \quad \Omega_T, \\ \partial_t \widetilde{V} = \Delta \widetilde{V} + a_{12}\widetilde{U} + a_{22}\widetilde{V} + \tilde{a}_{32}\widetilde{W} & \text{in} \quad \Omega_T, \\ \partial_t \widetilde{W} = \Delta \widetilde{W} + \tilde{a}_{13}\widetilde{U} + a_{23}\widetilde{V} + a_{33}\widetilde{W} & \text{in} \quad \Omega_T, \\ \widetilde{U} = k_1,\ \widetilde{V} = k_2,\ \widetilde{W} = k_3 & \text{on} \quad \Sigma_T, \\ \widetilde{U}(.,0) = U_0, \quad \widetilde{V}(.,0) = V_0 \quad \text{and} \quad \widetilde{W}(.,0) = W_0 & \text{in} \quad \Omega. \end{cases}$$

with $\widetilde{a_{ij}} \in C^4(\overline{\Omega})^9$. Following the method developped in [8] and [13], from Carleman estimate (3.4), we obtain the following identifiability and Lipschitz stability estimate for three coefficients (one in each equation) (e.g. a_{21}, a_{32}, a_{13}) by the observation of only one component on ω assuming the knowledge of these coefficients on ω for any subset ω of Ω.

THEOREM 4.4. *Assume that Assumption 3.2 is checked. Assume that there exists $\varepsilon > 0$ such that $k_1, k_2, k_3 \in H^1(0,T; H^{2+\varepsilon}(\partial\Omega)) \cap H^2(0,T; H^{\varepsilon}(\partial\Omega))$, $U_0, V_0, W_0 \in H^2(\Omega)$. Suppose that $(\widetilde{a_{ij}})$ are such that there exist $C > 0$ and $T' \in (0,T)$ such that $|\widetilde{U}(.,T')| \geq C$ in Ω, $|\widetilde{V}(.,T')| \geq C$ in Ω, $|\widetilde{W}(.,T')| \geq C$ in Ω. Assume that $a_{ij} = \widetilde{a_{ij}}$ on ω for $(i,j) \in \{(2,1),(3,2),(1,3)\}$.*
Then there exists $\kappa > 0$ such that

$$\|a_{21} - \tilde{a}_{21}\|^2_{L^2(\Omega)} + \|a_{32} - \tilde{a}_{32}\|^2_{L^2(\Omega)} + \|a_{13} - \tilde{a}_{13}\|^2_{L^2(\Omega)} \leq \kappa \Big(\|\partial_t(U - \widetilde{U})\|^2_{L^2(\omega_T)}$$
$$+\|(U - \widetilde{U})(T')\|^2_{H^2(\Omega)} + \|(V - \widetilde{V})(T')\|^2_{H^2(\Omega)} + \|(W - \widetilde{W})(T')\|^2_{H^2(\Omega)} \Big). \tag{4.6}$$

As for subsection 4.2, the previous result can be extended to the identification of n coefficients by $(n-2)$ local observations for a $(n \times n)$ reaction-diffusion system. We need only the knowledge of 3 coefficients on ω. Indeed, following the previous method for a (3×3) system, consider a $(n \times n)$ reaction-diffusion system like (4.2) and fix three components (e.g. y_1, y_2, y_3), the three first associated equations and three coefficients inside to recover. Assuming the knowledge of these three coefficients on the set of observation ω and using the previous Carleman estimate, we derive easily a stability estimate similar to (4.6) for n coefficients, one in each equation, by the observation of only $n-2$ components (y_1, and $(y_i)_{i\geq 3}$) without anymore assumption on the coefficients to recover.
All the regularity assumptions of system (4.2) can be weakened at it has been pointed out in Remark 3.5.

5. Comments. In this paper we presented only one result for a non linear case of coupled equations, Theorem 4.1. In this result we assumed that F was globally Lipschitz with respect to all its components. We think that this condition can be weakened assuming some slightly superlinear growth as in [15]. However, even if it is possible to obtain Carleman estimates with the explicit dependence of the L^∞ norms of the terms a_{1k}, we think that new technical difficulties may arose. To obtain the superlinear grow result it is necessary to construct more regular solutions, and if this can be done, clearly is a result out of the scope of this paper. It is worth mentioning that in the case of cascade systems of two equations, this was done in [19]. Also, for a particular non linear growing, in the case of two equations, Coron's return method was used in [12].

There are several open problems related with the controllability of *scalar* coupled linear parabolic equations and of course related with other parabolic systems that arise in other contexts as fluid dynamics. In this sense it is worth mentioning that there is a vast literature in this subject and the inherent problems are of different nature than the ones treated in this paper. To read about this interesting subject see e.g. [17], [11] and the references therein.

On the other hand it is interesting to point out that the condition $\partial\Omega \cap \partial\omega = \gamma$ with $|\gamma| \neq 0$ is used *only* to invert an operator of the form $B \cdot \nabla + ...$ We think that there my exist other conditions that warranty the invertibility of this operator.

General results for controllability of reaction-diffusion-advection systems of more that two equations are, as far as we know, open problems. Also to give necessary conditions or new sufficient conditions for the controllability of $n \times n$ systems for $n > 3$ is a challenging problem where even small progress is difficult.

6. Appendix.

6.1. The case of a general vector field B. Let us briefly show how one can transform (2.9) to the form (2.15). Let $(F, \mathcal{O})$ a C^2-parameterization of γ_T (or a subset of γ_T if necessary):

$$F : \left\{ \begin{array}{c} \mathcal{O} \mapsto \gamma_T \\ (y_1, ..., y_n) \mapsto (\sigma, t) \end{array} \right. .$$

Let $\tilde{B} = (B, 0)$. From Assumption 2.5, there exists $\Xi > 0$ such that the following map is well defined:

$$\Lambda : \begin{array}{ccc} (0, \Xi) \times \mathcal{O} & \to & \Omega_T \\ (\zeta, y) & \mapsto & \Lambda(\zeta, y), \end{array}$$

where, for all $y \in \mathcal{O}$,

$$\frac{d\Lambda}{d\zeta}(\zeta, y) = \tilde{B}(\Lambda(\zeta, y)), \quad \Lambda(0, y) = F(y)$$

and $\omega_{1,T} := \Lambda((0, \Xi) \times \mathcal{O}) \subset \omega_T$.
One has: $\partial\omega_{1,T} \cap \partial\Omega_T = \partial\omega_T \cap \partial\Omega_T = \gamma_T$ (possibly choosing a subset of γ_T).
By construction, we have $\partial_\zeta \Lambda = \tilde{B} \circ \Lambda$.
Under Assumption 2.5, the map Λ is onto from $(0, \Xi) \times \mathcal{O}$ to $\omega_{1,T}$, $\Lambda \in W^{2,\infty}(\omega_{1,T})$ (cf item 3) and $\Lambda^{-1} \in W^{2,\infty}((0, \Xi) \times \mathcal{O})$ (cf item 2).
Moreover, if we set $\widetilde{v} = v \circ \Lambda$, then $\partial_\zeta \widetilde{v} = (\ B \cdot \nabla v) \circ \Lambda$.
Let $(L, D(L))$ defined by $Lv = B \cdot \nabla v + bv, \quad D(L) = \{v \in H^1(\omega_{1,T})\, ;\, v|_{\gamma_T} = 0\}$.
By the change of variables, L is transformed in

$$\widetilde{L}\widetilde{v} = \partial_\zeta \widetilde{v} + \widetilde{b}\widetilde{v}, \quad D(\widetilde{L}) = \{\widetilde{v} \in H^1((0, \Xi) \times \mathcal{O})\, ;\, v(0, y) = 0, \quad \forall y \in \mathcal{O}\}, \tag{6.1}$$

with $\widetilde{b} = b \circ \Lambda$. Let

$$Lv = \partial_t u - div(H_1 \nabla u) - au - A \cdot \nabla u - f, \quad v \in D(L) \tag{6.2}$$

and denote $\widetilde{u} = u \circ \Lambda$, $\widetilde{f} = f \circ \Lambda$.

Then, with the same change of variables and under Assumption 2.5 there exist a $n \times n$ matrix $H = (h_{i,j}) \in W^{2,\infty}(\omega_T)^{n^2}$, a vector field $E = (E_i) \in W^{1,\infty}(\omega_T)^n$, a scalar field $e \in L^\infty(\omega_T)$ such that (6.2) is transformed into

$$\widetilde{L}\widetilde{v} = \partial_t \widetilde{u} - div(H\nabla\widetilde{u}) + E \cdot \nabla\widetilde{u} + e\widetilde{u} - \widetilde{f}, \quad \widetilde{v} \in D(\widetilde{L}). \tag{6.3}$$

The equation (6.3) with $\widetilde{L}$ defined by (6.1) has the form (2.15).
Moreover, in order to obtain (2.14) for a more general vector field B, note that

$$\iiint_{(0,\Xi)\times\mathcal{O}} ((s\rho)^{\tau_2+3}\xi e^{-2s\eta} \circ \Lambda)|\widetilde{v}|^2|Jac(\Lambda)|d\zeta dy = \iint_{\Omega_T} (s\rho)^{\tau_2+3}\xi e^{-2s\eta}|v|^2 dxdt, \tag{6.4}$$

where $Jac(\Lambda)$ denotes the Jacobian of Λ and ξ the cut-off function defined in (2.18). Now, we apply $\widetilde{L}^{-1}$ to (6.3) and obtain a relation for $\widetilde{v}$ similar to (2.17). We multiply each side by $((s\rho)^{\tau_2+3}\xi e^{-2s\eta}v) \circ \Lambda|Jac(\Lambda)|$, we integrate over $(0,\Xi)\times\mathcal{O}$ and using (6.4) we obtain (2.14) with

$$Qf(x,t) = (\widetilde{L}^{-1}\widetilde{f}) \circ \Lambda^{-1}(x,t), \tag{6.5}$$

with $\widetilde{L}$ and $\widetilde{f}$ defined previously.

6.2. Proof of Lemma 2.10. If $a \le 0$, the estimate (2.21) is satisfied for all $s, \lambda \ge 0$. Let us assume $a > 0$. Then:

$$\rho^a\, e^{-s\eta} \le 1 \Leftrightarrow e^{a\ln\rho - s\eta} \le 1 \Leftrightarrow a\ln\rho - s\eta < 0.$$

Let $\lambda_0 \ge \frac{\ln 2}{\beta}$ with $\beta^* := \max_{x\in\bar\Omega}\beta(x)$, then using (2.7) we have

$$\eta(x,t) \ge \rho^*, \quad \forall\lambda \ge \lambda_0, \quad \forall(x,t) \in \Omega_T.$$

So (2.21) holds if

$$a\ln\rho^* \le s\rho^*. \tag{6.6}$$

Using that for all $\lambda \ge \lambda_0$, one has

$$\rho^* \ge \frac{4e^{\lambda_0\beta_-}}{T^2} := \rho_0,$$

with $\beta_- := \min_{x\in\bar\Omega}\beta(x)$, we deduce that for all $s \ge \max\{\frac{aT^2}{4e^{\lambda_0\beta_-}}, \frac{a\ln\rho_0}{\rho_0}\} := s_0$ we have (6.6) and then (2.21) and the lemma is proved.

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